14-2 The Scalar (Dot) Product in 3D

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– 3 – N10/5/MATI

4. *[Maximum mark: 13]*

Solve the differential equation

$$(x-1)\frac{dy}{dx}+xy=(x-1)e^{-x}$$

given that $y=1$ when $x=0$. Give your answer in the form $y=f(x)$.

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4. writing the differential equation in standard form gives

$\frac{dy}{dx}+\frac{x}{x-1}y=e^{-x}$	*M1*
$\int\frac{x}{x-1}dx=\int\left(1+\frac{1}{x-1}\right)dx=x+\ln(x-1)$	*M1A1*
hence integrating factor is $e^{x+\ln(x-1)}=(x-1)e^{x}$	*M1A1*
hence, $(x-1)e^{x}\frac{dy}{dx}+xe^{x}y=x-1$	*(A1)*
$\Rightarrow\frac{d\left[(x-1)e^{x}y\right]}{dx}=x-1$	*(A1)*
$\Rightarrow(x-1)e^{x}y=\int(x-1)dx$	*A1*
$\Rightarrow(x-1)e^{x}y=\frac{x^2}{2}-x+c$	*A1*
substituting $(0, 1)$, $c=-1$	*(M1)A1*
$\Rightarrow(x-1)e^{x}y=\frac{x^2-2x-2}{2}$	*(A1)*
hence, $y=\frac{x^2-2x-2}{2(x-1)e^{x}}$ (or equivalent)	*A1*

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[13 marks]

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Let $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$,

then $\vec{u}\cdot\vec{v}=u_1\cdot v_1+u_2\cdot v_2+u_3\cdot v_3$

$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\theta$

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Special Cases

- If $\theta=0$, then
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\theta$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos 0$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cdot 1$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|$$
- If $\theta=\frac{\pi}{2}$, then
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\theta$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\left(\frac{\pi}{2}\right)$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cdot 0$$
$$\vec{u}\cdot\vec{v}=0$$
- If $\theta=\pi$, then
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\theta$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\pi$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cdot{}^{-}1$$
$$\vec{u}\cdot\vec{v}=-|u|\cdot|v|$$

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Special Cases Of Dot Products

If $\theta=0$, then

$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos\theta$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|\cos 0°$$
$$\vec{u}\cdot\vec{v}=|u|\cdot|v|$$

This is the largest possible value of the dot product.

If $\theta=\frac{\pi}{2}$, then

$$\vec{u}\cdot\vec{v}=|u_1|\cdot|v_1|\cos\theta$$
$$\vec{u}\cdot\vec{v}=|u_1|\cdot|v_1|\cos\frac{\pi}{2}$$
$$\vec{u}\cdot\vec{v}=|u_1|\cdot|v_1|\cdot 0$$
$$\vec{u}\cdot\vec{v}=0$$
$$\vec{u}\cdot\vec{v}=0 \qquad \vec{u}\perp\vec{v}$$

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If $\theta = \pi$, then

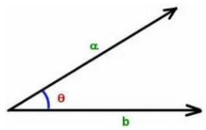
$\vec{u}\cdot\vec{v} = |u|\cdot|v|\cos\theta$

$\vec{u}\cdot\vec{v} = |u|\cdot|v|\cos\pi$

$\vec{u}\cdot\vec{v} = |u|\cdot|v|\cdot -1$

$\vec{u}\cdot\vec{v} = -|u|\cdot|v|$

This is the smallest possible value of the dot product.



$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

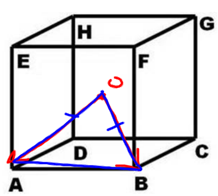
$\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right)$

Ex1. Find the angle between $\vec{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\vec{b} = -2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

$|\vec{a}| = \sqrt{14}$ $\quad |\vec{b}| = \sqrt{30}$

$\theta = \cos^{-1}\left(\dfrac{-6+5+-2}{\sqrt{14}\sqrt{30}}\right) = 98.4^\circ$

Ex2. Consider the unit cube ABCDEFGH. Let O be the point where the 4 diagonals meet.



$|\overline{AB}| = \sqrt{1} = 1$

$|\overline{AF}| = \sqrt{2}$

$|\overline{AG}| = \sqrt{3}$

$|\overline{OA}| = \frac{1}{2}\sqrt{3}$

$\overline{OA}\cdot\overline{OB} =$

$\angle AOB = 70.5^\circ$

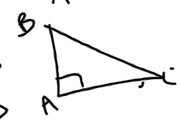
Ex3. A(4,5,-2), B(7,9,-3), C(6,4,0). Show that ΔABC is a right triangle and find its area.

$\overrightarrow{AB}\cdot\overrightarrow{CA} = -6+4+2 = 0$

$\overrightarrow{AB} = \langle 3,4,-1\rangle$

$\overrightarrow{BC} = \langle -1,-5,3\rangle$

$\overrightarrow{CA} = \langle -2,1,-2\rangle$

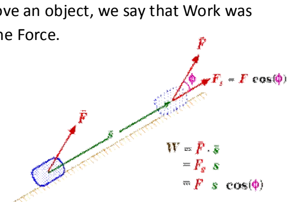


$A = \frac{1}{2}\left(\sqrt{9+16+1}\right)\left(\sqrt{4+1+4}\right)\sin 90^\circ$

$A = \dfrac{3\sqrt{26}}{2}$

Work

When a force acts to move an object, we say that Work was done on the object by the Force.



$\vec{F_i}$ = Force vector applied to the object/system.

$\vec{F}$ = Component of Force in the direction of movement.

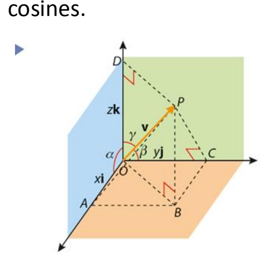
$\vec{s}$ = Displacement vector.

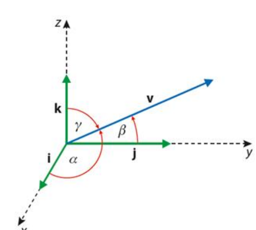
s = Distance the system is displaced.

ϕ = Angle between the displacement and the force.

Direction Cosines

The angles α, β, and γ that the vector makes with the unit coordinate vectors are called direction angles of $\vec{v}$ an $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the direction cosines.





Remember from basic triangle trig:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \Leftrightarrow \text{adjacent} = \text{hypotenuse} \cdot \cos\theta$$

So, in this case, we have:

$x = |\mathbf{v}|\cos\alpha$, $y = |\mathbf{v}|\cos\beta$, $z = |\mathbf{v}|\cos\gamma$, and so

$$\mathbf{v} = (|\mathbf{v}|\cos\alpha)\,\mathbf{i} + (|\mathbf{v}|\cos\beta)\,\mathbf{j} + (|\mathbf{v}|\cos\gamma)\,\mathbf{k} = |\mathbf{v}|\underbrace{(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})}.$$

Taking the magnitude of both sides, we have:

$$|\mathbf{v}| = |\mathbf{v}|\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}.$$

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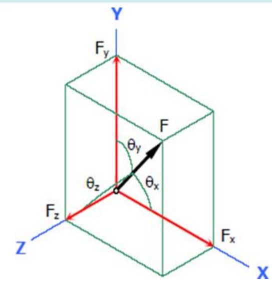
Therefore,
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, i.e. the sum of the squares of the direction cosines is always 1.
For a unit vector, the expression will be of the form

$$\mathbf{u} = |\mathbf{u}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}) = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}\ (|\mathbf{u}| = 1).$$

This means that for a unit vector its x-, y- and z-coordinates are its direction cosines.



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Ex4. Find the direction cosines for the vector v = 12i – 5j + 13k and then approximate the direction angles to the nearest degree.

ex) find direction cosines for $\vec{v} = 12i - 5j + 13k$ and then approximate the direction angles to the nearest degree.

$$\frac{12i - 5j + 13k}{\sqrt{144+25+169}} \quad \frac{12}{\sqrt{338}}i - \frac{5}{\sqrt{338}}j + \frac{13}{\sqrt{338}}k$$

$\cos\alpha = 12/\sqrt{338}$, $\alpha \approx 49°$

$\cos\beta = -5/\sqrt{338}$, $\beta \approx 106$

$\cos\gamma = 13/\sqrt{338}$, $\gamma \approx 45°$

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